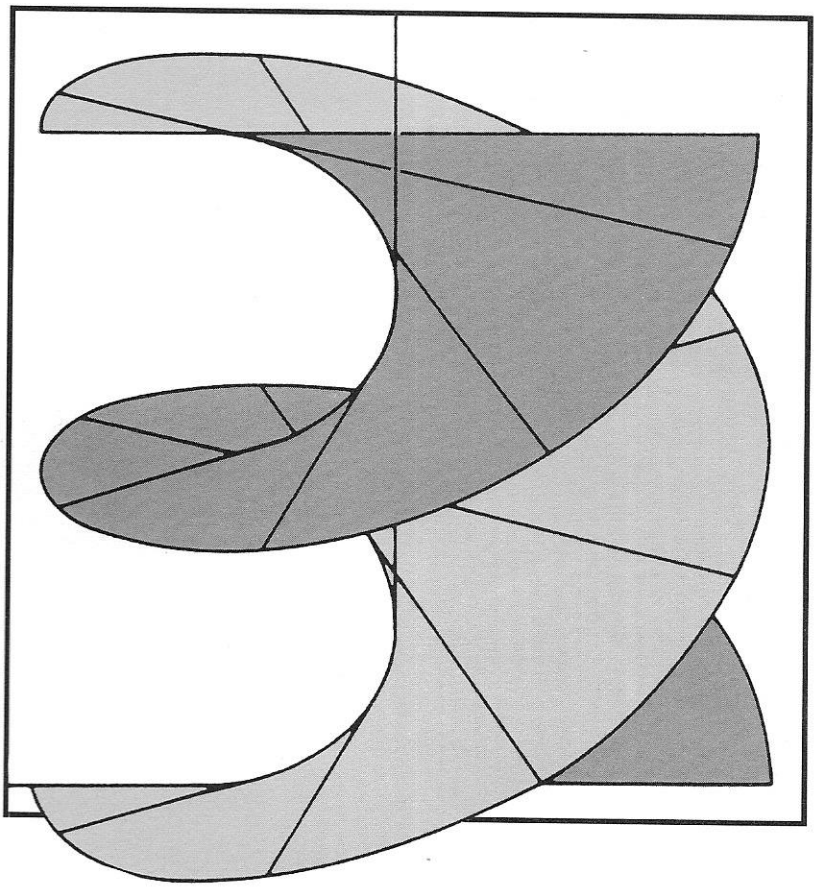
M434
DIFFERENTIAL GEOMETRY



PART III
EUCLIDEAN GEOMETRY

The Open University

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Mathematics: A Fourth Level Course

M434 Differential Geometry

Part III Euclidean Geometry

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

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Introduction

This part rounds off the study of curves as such, although curves will play an important role in the study of surfaces. The main aim of Chapter III of *O'Neill* is to prove a theorem to the effect that the curvature and torsion functions of a unit-speed curve completely determine the curve, except for the precise position of the route of the curve in $\mathbf{E}^3$.

In order to state the theorem precisely, we have to define exactly what is meant by two curves being 'the same'. The notion that we need is that of an **isometry** of $\mathbf{E}^3$.

We shall actually define what we mean by the routes being the same.

The first four sections are devoted to isometries. Section 1 defines isometries, links them to ideas of linear algebra and then shows that all isometries can be expressed in a form that makes computations easy.

Section 2 investigates the derivative maps of isometries. Since we shall be dealing with curves related by an isometry, we need to understand how isometries act on tangent vectors defined on the curves; this is precisely what the derivative map tells us.

In Section 3 we tackle the problem of distinguishing between left-handed and right-handed objects. The concept that we require is **orientation**. The need for such a concept can be illustrated by considering a circular helix, say, and its reflection in the xy-plane plane. In all respects except the 'handedness' of the spiral, the two helices are the same. The process of reflection (an isometry) is different in some essential respect from, say, translation (another isometry).

With the ideas from Sections 1–3, it remains to find out only how isometries affect acceleration of curves. We know that, in general, mappings do *not* preserve acceleration. However, isometries do and this will be important for the theorem that we are aiming to prove.

This was tackled in Chapter I.

Finally, Section 5 puts together the ideas about curves embodied in the Frenet apparatus and formulas with the work on isometries.

It may not always be apparent where the detailed work on isometries is leading. It may help if you remember that all the information about a curve α is contained in α and its first three derivatives. So, to understand the relationship between curves and isometries, we must be able to describe isometries and to understand what isometries (or, to be precise, their derivative maps) do to the derivatives of a curve.

We have formulas for the Frenet apparatus in terms of α and its first three derivatives.

Study advice

The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter III, Sections 1–3.

Week 2 *O'Neill*, Chapter III, Sections 4–5, the summary and TMA02.

1 Isometries

Read O'Neill: Chapter III, Section 1, pages 98–102.

Erratum *O'Neill*, page 102, the last line of text should read:

'Alternatively, ..., $q = F(p) \ldots$'. ■

This section has two aims.

(a) To define what is meant by an isometry.

(b) To show that any isometry can be expressed as an orthogonal transformation followed by a translation.

The second aim is important because it makes calculations with isometries very easy. It also makes the construction of an isometry to do a particular task a relatively straightforward process.

Isometries The mathematical concept of isometry is intended to formalize the idea of a *rigid motion*, a motion which changes the position of objects but not their size or shape. In particular, the distance between two points after the rigid motion must be the same as the distance between them before it. This property of preserving distance is taken as the defining property of isometries.

Many of the transformations that you have met before are isometries. Intuitively, it is fairly clear that rotations, reflections and translations are isometries. *O'Neill* deals explicitly with translations and rotations in his examples. The 'straightforward computation' referred to in the discussion of rotations is elementary, but messy, algebra. The result is the important thing, not the details of the justification.

M101 and *M203*

Composites and inverses Having defined a new type of object, isometries, you should be expecting the next steps. We show that the composite of isometries is also an isometry and that the inverse of an isometry is also an isometry.

The set of isometries is closed under composition and taking inverses.

Actually, *O'Neill* deals only with composition (in Lemma 1.3). *O'Neill* also observes that the inverse of a translation is a translation and, therefore, an isometry. The general result for inverses is not difficult and we include the proof here for completeness.

The idea of the proof is straightforward: because an isometry F preserves the distance between any two points of $\mathbf{E}^3$, in particular it will preserve the distance between $F^{-1}(\mathbf{p})$ and $F^{-1}(\mathbf{q})$.

Lemma 1.3a If F is an isometry of $\mathbf{E}^3$, then F^{-1} is also an isometry of $\mathbf{E}^3$.

Proof Suppose that $\mathbf{p}$ and $\mathbf{q}$ are points in $\mathbf{E}^3$. Then,

$$\begin{aligned} d(F^{-1}(\mathbf{p}), F^{-1}(\mathbf{q})) &= d(F(F^{-1}(\mathbf{p})), F(F^{-1}(\mathbf{q}))) \quad \text{(because } F \text{ is an isometry)} \\ &= d(\mathbf{p}, \mathbf{q}) \quad \text{(because } FF^{-1} \text{ is the identity).} \end{aligned}$$

Thus

$$d(F^{-1}(\mathbf{p}), F^{-1}(\mathbf{q})) = d(\mathbf{p}, \mathbf{q}),$$

and F^{-1} is an isometry.

Orthogonal transformation *O'Neill* defines this, in passing, as a linear transformation which preserves dot products. Since angles are, essentially, defined in terms of dot products by

$$\begin{aligned} \cos\theta &= \frac{\mathbf{p} \cdot \mathbf{q}}{\|\mathbf{p}\|\,\|\mathbf{q}\|} \\ &= \frac{\mathbf{p} \cdot \mathbf{q}}{\sqrt{\mathbf{p} \cdot \mathbf{p}}\sqrt{\mathbf{q} \cdot \mathbf{q}}}, \end{aligned}$$

orthogonal transformations preserve angles as well as dot products. In particular, they preserve right angles, hence the name *orthogonal.*

The plan of campaign for the proof of Theorem 1.7 can be set out as follows.

(a) Deal with isometries that fix the origin by;

- (i) showing that such an isometry preserves dot products;
- (ii) showing such an isometry is linear;
- (iii) deducing that such an isometry is an orthogonal transformation.

(a) Deal with the general case by combining it with a translation that ensures that the composite fixes the origin and, hence, that the composite is an orthogonal transformation.

Lemma 1.6 Two points arise here: polarization and linearity.

Polarization is the sort of idea that is 'obvious' once you have seen it! Actually, it simply involves careful application of the information about F and the definitions of norm and distance in terms of dot product.

The final steps in the linearity argument are omitted by *O'Neill* for the very good reason that they take up a lot of space for little insight. However, for completeness, here is the missing detail.

Suppose that

$$\begin{aligned}\mathbf{p} &= (p_1, p_2, p_3),\\ \mathbf{q} &= (q_1, q_2, q_3),\\ F(\mathbf{u}_i) &= \mathbf{v}_i, \quad i = 1, 2, 3.\end{aligned}$$

Then

$$\begin{aligned}F(a\mathbf{p} + b\mathbf{q}) &= F\left(\sum_{i=1}^{3}(ap_i\mathbf{u}_i + bq_i\mathbf{u}_i)\right)\\ &= F\left(\sum_{i=1}^{3}(ap_i + bq_i)\mathbf{u}_i\right)\\ &= \sum_{i=1}^{3}(ap_i + bq_i)F(\mathbf{u}_i)\\ &\quad \text{(because we know that } F \text{ acts linearly on the } \mathbf{u}_i\text{)}\\ &= \sum_{i=1}^{3}(ap_i + bq_i)\mathbf{v}_i\\ &= \sum_{i=1}^{3}(ap_i\mathbf{v}_i + bq_i\mathbf{v}_i)\\ &= a\sum_{i=1}^{3} p_i\mathbf{v}_i + b\sum_{i=1}^{3} q_i\mathbf{v}_i\\ &= aF(\mathbf{p}) + bF(\mathbf{q}).\end{aligned}$$

Once you know that F behaves linearly with respect to a basis (the $\mathbf{u}_i$) then it must behave linearly overall.

In the next section we shall apply the ideas contained in the proof of Lemma 1.6 to obtain an explicit formula for the orthogonal transformation that takes one given orthonormal basis into another.

Optional The comments about closure and inverses, made earlier, may have reminded you of work on groups from previous courses. If time permits, you might like to look at Exercises 7 and 8 which discuss groups of isometries.

Exercise 1.1 *O'Neill*, page 103, Exercise 1. (This exercise is included to emphasize that the order of the orthogonal part and translation part of an isometry is important.)

Exercise 1.2 *O'Neill*, page 103, Exercise 2. (*Hint*: Apply the first exercise.)

Exercise 1.3 *O'Neill*, page 103, Exercise 4.

Exercise 1.4 *O'Neill*, page 103, Exercise 5.

Exercise 1.5 *O'Neill*, page 103, Exercise 6.

[*Solutions on page 16*]

2 Derivative maps of isometries

Read *O'Neill: Chapter III, Section 2, pages 104–106.*

Having obtained the computationally useful expression

$$F = TC$$

for an isometry, we now go on to consider the derivative map of an isometry. In a way, the result of Theorem 2.1 is almost inevitable. The orthogonal part of an isometry determines (in a crude sense) how geometric objects will be 'rotated' whereas the translation part alters only their position in space. Because an isometry is a rigid motion, any collection of tangent vectors attached to an object will all be 'rotated' in the same way. Expressed precisely, that is exactly what Theorem 2.1 says.

In matrix form, the vector part of $F_*(\mathbf{v}_\mathbf{p})$ is given by

$$C \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Corollary 2.2 There is a further deduction that we can make. By applying the pointwise principle, we can say that

$$F_*(V) = C(V)$$

for any vector field on $\mathbf{E}^3$. Because F_* preserves dot products, we can deduce that

$$F_*(V) \cdot F_*(W) = V \cdot W$$

for any vector fields V and W on $\mathbf{E}^3$. It also follows that F_* maps frame fields to frame fields.

If E_1, E_2 and E_3 form a frame field, then

$$\begin{aligned} F_*(E_i) \cdot F_*(E_j) &= E_i \cdot E_j \\ &= \delta_{ij}. \end{aligned}$$

Hence $F_*(E_1)$, $F_*(E_2)$ and $F_*(E_3)$ form a frame field.

Theorem 2.3 With this theorem and the remark that follows, the main aim of this section has been achieved.

Given two points, each equipped with a frame, we can find the unique isometry that maps the first point-and-frame to the second.

Strictly, the isometry maps the point, its derivative map deals with the frame.

It may help to understand where the expression

$$C = {}^t\!B\,A$$

comes from if we look in a little more detail at exactly what is required of C.

The notation set up in Theorem 2.3 means that

$$\begin{aligned} e_1 &= (a_{11}, a_{12}, a_{13}) \\ e_2 &= (a_{21}, a_{22}, a_{23}) \\ e_3 &= (a_{31}, a_{32}, a_{33}) \\ f_1 &= (b_{11}, b_{12}, b_{13}) \\ f_2 &= (b_{21}, b_{22}, b_{23}) \\ f_3 &= (b_{31}, b_{32}, b_{33}). \end{aligned}$$

We want $C(e_i) = f_i$ so we want C to have the following effect.

$$\begin{aligned} C\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} &= \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} \\ C\begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} &= \begin{pmatrix} b_{21} \\ b_{22} \\ b_{23} \end{pmatrix} \\ C\begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} &= \begin{pmatrix} b_{31} \\ b_{32} \\ b_{33} \end{pmatrix}. \end{aligned}$$

If we combine these as a single matrix equation, we get

$$C\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}.$$

We can write this equation as

$$C\,{}^tA = {}^tB.$$

We can solve this for C by multiplying on the right by $({}^tA)^{-1}$ and, since A and ${}^tA$ are orthogonal, $({}^tA)^{-1} = {}^t({}^tA) = A$. Hence

$$\begin{aligned} C &= {}^tB\,{}^tA^{-1} \\ &= {}^tB\,A. \end{aligned}$$

A point easily overlooked is that the translation part of the isometry is

$$\mathbf{q} - C(\mathbf{p})$$

and *not* $\mathbf{q} - \mathbf{p}$. This is a consequence of the expression

$$F = TC$$

in which the orthogonal part is applied first. Overall, we require

$$F(\mathbf{p}) = \mathbf{q}$$

but

$$\begin{aligned} F(\mathbf{p}) &= (TC)(\mathbf{p}) \\ &= T(C(\mathbf{p})). \end{aligned}$$

Thus T is the translation taking $C(\mathbf{p})$ to $\mathbf{q}$.

We can now predict that, when an isometry is applied to a curve, the orthogonal part will determine what happens to the Frenet frame at each point of the curve.

In spite of the fact that applying Theorem 2.3 is a very important skill, we have set only one exercise on it. The reason is that the application to curves will give you further opportunity to develop this skill.

Exercise 2.1 *O'Neill*, page 106, Exercise 1.

Exercise 2.2 *O'Neill*, page 107, Exercise 5.

[*Solutions on page 18*]

3 Orientation

Read *O'Neill: Chapter III, Section 3, pages 107–110.*

Errata

1 *O'Neill*, page 108, the second displayed equation in the proof of Lemma 3.2 should read:

$$\left(\sum_k c_{jk} a_{ik}\right) = \left(\sum_k a_{ik} {}^t c_{kj}\right) = A\,{}^tC.$$

2 *O'Neill*, page 108, the final calculations in the proof of Lemma 3.2 should read:

$$\begin{aligned} F_*(\mathrm{e}_1) \cdot F_*(\mathrm{e}_2) \times F_*(\mathrm{e}_3) &= \det(A\,{}^tC) \\ &= \det A \det {}^tC \\ &= \det A \det C \\ &= \mathrm{sgn} F \mathrm{e}_1 \cdot \mathrm{e}_2 \times \mathrm{e}_3. \end{aligned}$$

■

Orientation The underlying idea behind the formal definition of orientation (left- and right-handed gloves) is probably not as elusive as *O'Neill* suggests. It is the formalization that is tricky.

Lemma 3.2 You may find the following, expanded, version of the proof helpful.

Let us denote the attitude matrix of the frame field $F_*(\mathrm{e}_i)$, $i = 1, 2, 3$ by B. Then, writing out the effect of F_*, that is C, in full, we get

$$\begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} = C \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}$$

$$\begin{pmatrix} b_{21} \\ b_{22} \\ b_{23} \end{pmatrix} = C \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix}$$

$$\begin{pmatrix} b_{31} \\ b_{32} \\ b_{33} \end{pmatrix} = C \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix}.$$

Combining these, we have

$${}^tB = C\,{}^tA.$$

Now, the triple scalar product

$$F_*(\mathrm{e}_1) \cdot F_*(\mathrm{e}_2) \times F_*(\mathrm{e}_3)$$

is the determinant of the attitude matrix B. But,

$$\begin{aligned} \det(B) &= \det({}^t(C\,{}^tA)) \\ &= \det({}^{tt}A\,{}^tC) \\ &= \det(A\,{}^tC) \\ &= \det(A)\ \det({}^tC) \\ &= \det(A)\ \det(C) \\ &= \mathrm{sgn}\, F\, \mathrm{e}_1 \cdot \mathrm{e}_2 \times \mathrm{e}_3. \end{aligned}$$

Exercise 3.1 On page 107, Exercise 5 of *O'Neill*, two frames are defined. Find their respective orientations.

In the last section you found an isometry, F, taking the first frame to the second. Find sgn F and hence verify Lemma 3.2 for this example.

Exercise 3.2 Let the orthogonal transformation C be specified by the matrix

$$\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

and let the vectors $\mathbf{v}$ and $\mathbf{w}$ be given by

$$\mathbf{v} = (3, 1, -1) \quad \text{and} \quad \mathbf{w} = (-3, -3, 1).$$

Check the formula

$$C(\mathbf{v} \times \mathbf{w}) = \operatorname{sgn} C(\mathbf{v}) \times C(\mathbf{w}).$$

[*Solutions on page 18*]

4 Euclidean geometry

Read *O'Neill: Chapter III, Section 4, pages 112–115.*

Erratum *O'Neill*, page 113, in Fig. 3.6, the vector attached to $\overline{\alpha}(t)$ should be labelled $\overline{Y}(t)$, *not* $Y(t)$. ■

This section provides the final technical tools that we shall need for our discussion of congruence of curves and the first half of the theorem that is the main aim of this part of the course.

The central theme of this section is finding out what happens when an isometry is applied to a curve. We already know how to find the effect of the isometry on the Frenet frame that is attached to each point on the curve: we apply the derivative map. For isometries, the derivative map has a particularly simple form. We also need to know about the relationship between the Frenet formulas for the two curves and these involve the *derivatives* of the Frenet frame. The main result of this section describes exactly how isometries affect derivatives of vector fields. This result is then applied to the Frenet frame of a unit-speed curve.

We know this from the general work on mappings.

Acceleration and mappings You might like to work through the compressed example in the middle of page 112 in a little more detail. We suggest that you ignore the remark about diffeomorphisms.

Corollary 4.1 It is not *quite* clear of what this is a corollary! Actually, it follows from the fact that the derivative map of an isometry is effected by the *same* matrix at each point of $\mathbf{E}^3$. The core of the proof is the paragraph beginning: 'On the other hand...'

Theorem 4.2 This is the application of Corollary 4.1 to curves.

In *Part II*, Section 4, we showed that all the Frenet apparatus of a curve α can be expressed in terms of the first three derivatives of α by using dot products, norms and cross products. These derivatives define three vector fields on the curve and Corollary 4.1 tells us that such vector fields are preserved when we apply an isometry to the curve. Isometries also preserve dot products, norms and (making allowances for sign) cross products.

Taking these facts into account, it is reasonable to suppose that applying an isometry to a curve gives a new curve with, essentially, the same Frenet apparatus as the old one. The theorem makes these ideas precise and then proves that the expected actually happens.

The proof makes use of some deductions from the definitions of the Frenet apparatus. Since β is unit speed, we have

$$\begin{aligned} T &= \beta', \\ \kappa &= \|T'\| = \|\beta''\|, \\ N &= \frac{T'}{\kappa} = \frac{\beta''}{\kappa}. \end{aligned}$$

When showing that $\overline{\kappa} = \kappa$, the proof uses the fact that F_* preserves dot products, and hence norms, in the line

$$\ldots \|F_*(\beta'')\| = \|\beta''\| \ldots.$$

In the calculation of $\overline{N}$, the proof uses the linearity of F_* by asserting that

$$\frac{F_*(\beta'')}{\kappa} = F_*\left(\frac{\beta''}{\kappa}\right).$$

Note, also, that *O'Neill* uses the modified definition of τ that we gave in *Part II*, Section 3.

That is $\tau = -B' \cdot N$.

Anticipating the beginning of the next section a little, we define two curves α and β to be congruent if there is an isometry F such that

$$F(\alpha) = \beta.$$

This theorem can then be restated in the following form.

If α and β are congruent unit-speed curves, then

$$\kappa_\alpha = \kappa_\beta \quad \text{and} \quad \tau_\alpha = \pm\tau_\beta.$$

The sign in the second equation is determined by the sign of the isometry which makes the two curves congruent.

The next section proves the converse: that if we know that two unit-speed curves have the same curvature and (making the allowance for sign) the same torsion, then the curves must be congruent. We shall show explicitly also how to produce the isometry to justify this assertion.

The two results together show that the curvature and torsion give a complete geometric description of a curve.

Non unit-speed curves The results in *O'Neill* on the effect of isometries on curves deal only with unit-speed curves. It is useful to add one further result.

Lemma 4.4 If α is a curve in $\mathbf{E}^3$ and F is an isometry, then

$$v_{F(\alpha)} = v_\alpha,$$

that is, F preserves speed.

Proof We know that F is a mapping and so preserves velocity, that is,

$$(F(\alpha))' = F_*(\alpha').$$

Now

$$\begin{aligned} v_{F(\alpha)} &= \|\,(F(\alpha))'\,\| \\ &= \|F_*(\alpha')\| \\ &= \sqrt{F_*(\alpha') \cdot F_*(\alpha')} \\ &= \sqrt{\alpha' \cdot \alpha'} \quad \text{(because isometries preserve dot products)} \\ &= \|\alpha'\| \\ &= v_\alpha. \end{aligned}$$

This completes the proof.

Exercise 4.1 *O'Neill*, page 115, Exercise 1.

Exercise 4.2 *O'Neill*, page 115, Exercise 2.

[*Solutions on page 19*]

5 Congruence of curves

Read *O'Neill: Chapter III, Section 5, pages 116–121.*

Erratum *O'Neill*, page 120, the last line of the proof of Corollary 5.6 should read:

$$F(\alpha(t)) = F(\overline{\alpha}(s(t))) = \overline{\beta}(s(t)) = \beta(t). \qquad \blacksquare$$

In this section we prove the converse of Theorem 4.2 to complete the proof of the following result.

Main theorem on curves If α and β are unit speed curves in $\mathbf{E}^3$ then α and β are congruent if, and only if,

$$\kappa_\beta = \kappa_\alpha \quad \text{and} \quad \tau_\beta = \pm\tau_\alpha.$$

A by-product of the method of proof is that it is possible to give an explicit recipe for constructing the isometry in such cases.

We also extend the theorem to non unit-speed curves. This extension requires the additional condition that

$$v_\beta = v_\alpha > 0.$$

The overall strategy is the following.

(a) Deal with the case in which the two curves are related by a translation.

(b) Deal with the general case, making use of the special case.

Let us consider two curves, one a translated version of the other. We know that, for a translation, the orthogonal part, C is represented by the identity matrix. Since it is C that determines what happens to the velocity vector of a curve, velocity vectors of the two curves will actually have the *same* Euclidean coordinate functions. Thus the velocity vectors will be parallel.

As usual, when discussing isometries, we use the $F = TC$ notation without comment.

Lemma 5.2 This lemma deals with the converse of the observation above. It shows that we can detect if two given curves are related by a translation by inspecting their velocities. If their velocity vector fields are parallel, there is a translation mapping one curve to the other.

Note that you cannot actually calculate the three constants $p_i, i = 1, 2, 3$ unless you know a pair of corresponding points on the two curves. Without this knowledge all that you can assert is the *existence* of the constants. Fortunately, this is sufficient for our purposes.

Theorem 5.3 The approach to the general case is mildly complicated by the need to consider the cases

$$\tau_\beta = \tau_\alpha \quad \text{and} \quad \tau_\beta = -\tau_\alpha$$

separately.

The basic idea is fairly straightforward. We choose corresponding points on each curve and choose an isometry that will accomplish the following:

- map $\alpha(0)$ to $\beta(0)$,
- map the Frenet frame at $\alpha(0)$ to the Frenet frame at $\beta(0)$, allowing for a small complication.

Because equality of the *functions* κ_α and κ_β implies that

$$\kappa_\alpha(0) = \kappa_\beta(0),$$

we choose $\alpha(0)$ and $\beta(0)$ as the pair of corresponding points.

We can always construct an isometry to map a given point to another and, simultaneously, map a chosen frame at the first point to a frame at the second.

This was the technique you practised in Section 2.

The 'small complication' referred to above is the fact that Frenet frames are, by definition, always positively oriented. We know that we need to consider orientation reversing isometries and so the isometry which simply maps one Frenet frame to the other will not always be quite what is needed..

We construct the isometry to map T_α and N_α to T_β and N_β respectively. We then arrange

$$\begin{aligned} B_\alpha &\longmapsto B_\beta \quad \text{if } \tau_\alpha = \tau_\beta, \\ B_\alpha &\longmapsto -B_\beta \quad \text{if } \tau_\alpha = -\tau_\beta. \end{aligned}$$

Intuitively, the isometry F, constructed to this recipe, ensures that $\overline{\alpha} = F(\alpha)$ and β coincide at $\beta(0)$ *and* set off in the same direction from $\beta(0)$. The final stage in the proof is to show that, not only do they set off in the same direction but they remain exactly the same. To do so, we show that their velocities are the same. This implies that they are related by a translation. However, the two curves coincide at $t = 0$, so the translation is the identity. That is an involved way of saying that

This is why we needed the special case first.

$$\overline{\alpha} = F(\alpha) = \beta.$$

The detail of the proof that the tangents are parallel,

$$\overline{T} = T$$

in the notation of the proof, gets a little involved. Let us look carefully at what *O'Neill* does.

The expression

$$\overline{T} \cdot T + \overline{N} \cdot N + \overline{B} \cdot B$$

contains three terms, each of which is the dot product of two unit vectors. Using the Schwarz inequality is rather a case of overkill. If $\overline{T}$ and T are at an angle θ then we know that

$$\begin{aligned} \overline{T} \cdot T &= \|\overline{T}\|\,\|T\| \cos\theta \\ &\leq \|\overline{T}\|\,\|T\| \quad (\text{because } -1 \leq \cos\theta \leq +1) \\ &= 1 \quad (\text{both are unit vectors}). \end{aligned}$$

Equality will occur only when $\cos\theta = 1$ and the vectors are equal.

Two similar results hold for the other two terms and we deduce that

$$\overline{T} \cdot T + \overline{N} \cdot N + \overline{B} \cdot B \leq 3$$

with equality only when corresponding vectors are equal.

Having established that the function

$$f = \overline{T} \cdot T + \overline{N} \cdot N + \overline{B} \cdot B$$

has the value 3 at one point, namely $t = 0$, we go on to show that its derivative is zero, so that f is constant. It is at this point that the Frenet formulas are used. The details omitted by *O'Neill* are as follows.

$$\begin{aligned} f' &= \overline{T}' \cdot T + \overline{T} \cdot T' + \overline{N}' \cdot N + \overline{N} \cdot N' + \overline{B}' \cdot B + \overline{B} \cdot B' \\ &= \overline{\kappa N} \cdot T + \overline{T} \cdot \kappa N + (-\overline{\kappa T} + \overline{\tau B}) \cdot T + \overline{N} \cdot (-\kappa T + \tau B) - \overline{\tau N} \cdot B - \overline{B} \cdot \tau N \\ &= 0. \end{aligned}$$

It should be clear that the terms *do* cancel in pairs as *O'Neill* asserts.

Comment: You may well have been unhappy about writing down expressions such as

$$\overline{T} \cdot T$$

before we have proved that the two curves are the same. Until the proof is complete, we really cannot be sure that $\overline{T}$ and T have the same point of application. You are intended to interpret these expressions as the result of taking the dot products after moving both vectors to the origin.

If you did not spot this inconsistency, do not worry! The comment is just in case you did.

Example 5.4 Here *O'Neill* points out the application of the method of Section 2 to finding the isometry required to map one curve to another.

Corollary 5.5 We already have a test for deciding whether or not a curve is a *cylindrical* helix: calculate the ratio τ/κ. The curve is a cylindrical helix if, and only if, this ratio is a constant.

What *O'Neill* is discussing here are *circular* helices.

Corollary 5.6 This corollary removes the restriction to unit-speed curves.

Theorem 5.7 *O'Neill* presents this for completeness and because of the application to which he refers. Since the part of *O'Neill* referred to does not form part of this course, you may omit this theorem, or simply read it out of interest.

The achievements of this section are substantial. The Frenet apparatus gives a standard test for the congruence of curves. Moreover, if two curves are found to be congruent, then we have a method of constructing the isometry which makes them congruent.

In constructing such isometries, the only point that needs to be watched is what happens to B. Symbolically

$$\left.\begin{matrix} T_\alpha(0) \\ N_\alpha(0) \\ B_\alpha(0) \end{matrix}\right\} \longmapsto \left\{\begin{matrix} T_\beta(0) \\ N_\beta(0) \\ \pm B_\beta(0) \end{matrix}\right.$$

with the sign chosen to match the sign in $\tau_\beta = \pm\tau_\alpha$.

The first two exercises give you a chance to practise the basic techniques resulting from Theorem 5.3. The others develop the ideas slightly and may be omitted if you are short of time.

Exercise 5.1 The curves α and β are defined by

$$\alpha(t) = (\cos t, \sin t, 0),$$
$$\beta(t) = \tfrac{1}{3}(-2\cos t + 2\sin t + 3, 2\cos t + \sin t - 3, \cos t + 2\sin t).$$

Show that α and β are congruent and find an isometry F such that

$$\beta = F(\alpha).$$

Exercise 5.2 The curves α and β are defined by

$$\alpha(t) = (3t - t^3, 3t^2, 3t + t^3),$$
$$\beta(t) = (3\sqrt{2}t, -3t^2, \sqrt{2}t^3).$$

Show that α and β are congruent and find an isometry F such that

$$\beta = F(\alpha).$$

(*Hint*: To save you some time, note that α was discussed in Chapter II, Example 4.4 on pages 69–70 of *O'Neill*. Use the results obtained there to shorten your working.)

Exercise 5.3 *O'Neill*, page 121, Exercise 1.

Exercise 5.4 *O'Neill*, page 122, Exercise 4.

[*Solutions on page 20*]

6 Summary

Read *O'Neill: Chapter III, Section 6, page 123.*

This chapter has introduced the formal definitions necessary to give a definition of congruence of curves in $\mathbf{E}^3$.

Using these ideas it was possible to prove that the speed, curvature and torsion functions completely determine a curve in $\mathbf{E}^3$, except for the position of its route in space.

The techniques which have been developed from the theory are:

(a) deciding whether or not two curves are congruent;

(b) when two curves *are* congruent, constructing an isometry that maps one to the other.

This part of the course completes the direct study of curves. In the next part we begin the study of surfaces; some of our results about curves will be useful tools in this study.

Solutions to the exercises

Solution 1.1

In order to prove that two transformations (i.e. functions) are the same, we show that they have the same effect on a typical member of the domain. So, let $\mathbf{p} \in \mathbf{E}^3$. Then

$$\begin{aligned}(CT_a)(\mathbf{p}) &= C(T_a(\mathbf{p}))\\ &= C(\mathbf{p}+\mathbf{a}) \quad (\text{definition of } T_a)\\ &= C(\mathbf{p}) + C(\mathbf{a}) \quad (C \text{ is linear})\\ &= T_{C(a)}(C(\mathbf{p})) \quad (\text{adding } C(a) \text{ is } T_{C(a)})\\ &= (T_{C(a)}C)(\mathbf{p}).\end{aligned}$$

Hence

$$CT_a = T_{C(a)}C.$$

Solution 1.2

As suggested in the hint, we apply the above result.

$$\begin{aligned}FG &= (T_aA)(T_bB)\\ &= T_a(AT_b)B\\ &= T_a(T_{A(b)}A)B \quad (\text{using the previous solution})\\ &= (T_aT_{A(b)})(AB)\\ &= T_{a+A(b)}(AB).\end{aligned}$$

Hence the translation part of FG is

$$T_{a+A(b)},$$

and the orthogonal part is

$$AB,$$

the composite of the orthogonal parts of F and of G.

Note that the translation part is not simply the composite of the separate translation parts.

To deal with GF, we could repeat the entire calculation. However, it is easier just to interchange a with b, A with B. Thus, the translation part of GF is

$$T_{b+B(a)}$$

and the orthogonal part is

$$BA.$$

Solution 1.3

The most straightforward check that C is orthogonal is to calculate ${}^tC\,C$.

$$\begin{aligned}{}^tC\,C &= \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

Now, in matrix form,

$$\begin{aligned}C(\mathbf{p}) &= \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}3\\1\\-6\end{pmatrix}\\ &= \begin{pmatrix}\frac{2}{3}\\ \frac{19}{3}\\ -\frac{7}{3}\end{pmatrix}.\end{aligned}$$

$$\begin{aligned}C(\mathbf{q}) &= \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1\\0\\3\end{pmatrix}\\ &= \begin{pmatrix}-\frac{5}{3}\\ -\frac{4}{3}\\ \frac{7}{3}\end{pmatrix}.\end{aligned}$$

Now,

$$\begin{aligned}C(\mathbf{p})\cdot C(\mathbf{q}) &= -\frac{2}{3}\frac{5}{3} - \frac{19}{3}\frac{4}{3} - \frac{7}{3}\frac{7}{3}\\ &= -\frac{135}{9}\\ &= -15.\\ \mathbf{p}\cdot\mathbf{q} &= 3\times 1 + 1\times 0 - 6\times 3\\ &= -15\\ &= C(\mathbf{p})\cdot C(\mathbf{q}).\end{aligned}$$

Solution 1.4

(a) Since

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2\\-2\\8\end{pmatrix} = \begin{pmatrix}-3\sqrt{2}\\-2\\5\sqrt{2}\end{pmatrix},$$

we have

$$C(\mathbf{p}) = (-3\sqrt{2}, -2, 5\sqrt{2}).$$

Hence

$$\begin{aligned}\mathbf{q} &= F(\mathbf{p})\\ &= (T_aC)(\mathbf{p})\\ &= T_a((-3\sqrt{2}, -2, 5\sqrt{2}))\\ &= (-3\sqrt{2}, -2, 5\sqrt{2}) + (1, 3, -1)\\ &= (-3\sqrt{2}+1, 1, 5\sqrt{2}-1).\end{aligned}$$

(b) For this part, we can usefully apply the solution to the first exercise to get an explicit expression for F^{-1}.

$$\begin{aligned}F^{-1} &= (T_aC)^{-1}\\ &= C^{-1}T_a^{-1}\\ &= {}^tC\,T_{-a}\\ &= T_{{}^tC(-a)}\,{}^tC.\end{aligned}$$

We now calculate the image of $\mathbf{p}$ in exactly the same way as above. Firstly,

$$
{}^tC(\mathbf{p}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix}
$$

$$
= \begin{pmatrix} 5\sqrt{2} \\ -2 \\ 3\sqrt{2} \end{pmatrix},
$$

we have

$$
{}^tC(\mathbf{p}) = (5\sqrt{2}, -2, 3\sqrt{2}).
$$

Now

$$
-\mathbf{a} = (-1, -3, 1),
$$

so

$$
{}^tC(-\mathbf{a}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ -3 \\ \sqrt{2} \end{pmatrix}.
$$

Thus

$$
\begin{aligned}
\mathbf{q} &= F^{-1}(\mathbf{p}) \\
&= (5\sqrt{2}, -5, 4\sqrt{2}).
\end{aligned}
$$

(c) We have

$$
T_a(\mathbf{p}) = (3, 1, 7)
$$

and

$$
\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} \\ 1 \\ 5\sqrt{2} \end{pmatrix}.
$$

Thus

$$
\mathbf{q} = (CT_a)(\mathbf{p}) = (-2\sqrt{2}, 1, 5\sqrt{2}).
$$

Solution 1.5

There are several possible approaches to this question. Perhaps the most obvious is to begin by using the definition of isometry as a test: check if

$$
\|F(\mathbf{p}) - F(\mathbf{q})\| = \|\mathbf{p} - \mathbf{q}\|.
$$

Alternatively, one could try to express each transformation in the form

$$
T_aC,
$$

where C is *some* matrix transformation and then check C for orthogonality.

On the whole we shall adopt the 'first principles' approach. The problem with this is that if F is not an isometry, it is sometimes difficult to see why not.

(a) Here we have

$$
\begin{aligned}
\|F(\mathbf{p}) - F(\mathbf{q})\| &= \| - \mathbf{p} - (-\mathbf{q})\| \\
&= \|\mathbf{q} - \mathbf{p}\| \\
&= \|\mathbf{p} - \mathbf{q}\|.
\end{aligned}
$$

Thus F is an isometry.

(b) In this case

$$
\begin{aligned}
\|F(\mathbf{p}) - F(\mathbf{q})\| &= \|\mathbf{p} \cdot \mathbf{a}\,\mathbf{a} - \mathbf{q} \cdot \mathbf{a}\,\mathbf{a}\| \\
&= \|(\mathbf{p} - \mathbf{q}) \cdot \mathbf{a}\,\mathbf{a}\| \\
&= \|\mathbf{p} - \mathbf{q}\|\,\|\mathbf{a}\| \cos\theta\,\|\mathbf{a}\| \\
&\qquad \text{(where } \theta \text{ is the angle between } \mathbf{p} - \mathbf{q} \text{ and } \mathbf{a} \\
&= \|\mathbf{p} - \mathbf{q}\| \cos\theta \quad (\text{since } \|\mathbf{a}\| = 1).
\end{aligned}
$$

Since there is no reason why θ should be zero for all $\mathbf{p}$ and $\mathbf{q}$, F is not an isometry. This observation enables us to construct a counterexample; all we need to do is choose $\mathbf{p}$ and $\mathbf{q}$ so that $\theta \neq 0$. To do so, let $\mathbf{p}$ be any non-zero vector perpendicular to $\mathbf{a}$ and let $\mathbf{q}$ be the zero vector. Then $\mathbf{p} - \mathbf{q}$ is orthogonal to $\mathbf{a}$ and so $\cos\theta = 0$. Hence

$$
\|\mathbf{p}\| \neq 0
$$

but

$$
\|F(\mathbf{p})\| = \|\mathbf{0}\| = 0.
$$

(c) If we write

$$
F(\mathbf{p}) = (p_3, p_2, p_1) - (1, 2, 3),
$$

then we can see that $F = TC$, where T is translation by

$$
(-1, -2, -3),
$$

and C is defined by the matrix

$$
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Since C is orthogonal (it has determinant -1), F is an isometry.

A 'first principles' computation also shows directly that F is an isometry.

(d) If we compare $\|F(\mathbf{p}) - F(\mathbf{q})\|$ with $\|\mathbf{p} - \mathbf{q}\|$, we get

$$
\begin{aligned}
\|F(\mathbf{p}) - F(\mathbf{q})\| &= \|(p_1, p_2, 1) - (q_1, q_2, 1)\| \\
&= \|(p_1 - q_1, p_2 - q_2, 0)\| \\
&= \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \\
\|\mathbf{p} - \mathbf{q}\| &= \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}.
\end{aligned}
$$

We can now see that, if $p_3 - q_3 \neq 0$, these two results are different.

Hence F is not an isometry.

There is a directly geometric way of seeing that F cannot be an isometry. It maps the whole of $\mathbf{E}^3$ onto the *plane*

$$
z = 1.
$$

Therefore, some points which were separate in the domain have the same images in the codomain. Isometries have to be one–one, so F cannot be an isometry.

Solution 2.1

The best thing to do is probably to cast the translation into the standard form for isometries and apply the results of the section.

Informally, a translation has no orthogonal part. Formally, this means that a translation is an isometry for which

$$
C = I,
$$

the identity matrix. It follows that T_* is represented by the identity matrix.

It follows that the vector part of $T_*(\mathbf{v}_\mathbf{p})$ is

$$
I(\mathbf{v}) = \mathbf{v}.
$$

The result follows.

Solution 2.2

If we use the notation of the remark after Theorem 2.3, we have

$$A = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

$${}^tB = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$\begin{aligned} C &= {}^tB\,A \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}. \end{aligned}$$

Since

$$C(\mathbf{p}) = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{4}{3\sqrt{2}} \end{pmatrix},$$

we require translation by

$$\begin{aligned} \mathbf{q} - C(\mathbf{p}) &= \left(3 - 0, -1 - \frac{1}{3}, 1 - \frac{4}{3\sqrt{2}}\right) \\ &= \left(3, -\frac{4}{3}, 1 - \frac{4}{3\sqrt{2}}\right). \end{aligned}$$

Solution 3.1

The first frame has attitude matrix

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

and so its orientation is obtained from the determinant whose value is

$$\begin{aligned} &\frac{2}{3}\left(\frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{2}{3}\right) - \frac{2}{3}\left(-\frac{2}{3}\frac{2}{3} - \frac{2}{3}\frac{1}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\frac{2}{3} - \frac{1}{3}\frac{1}{3}\right) \\ &= \frac{1}{27}(2 \times 6 - 2 \times (-6) + 1 \times 3) \\ &= \frac{1}{27} \times 27 \\ &= 1. \end{aligned}$$

Thus the frame is positively oriented.

For the second frame, the attitude matrix is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

with determinant

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}} - 0\right) - 0 + \frac{1}{\sqrt{2}}\left(0 - \frac{1}{\sqrt{2}}\right) \\ &= -\frac{1}{2} - \frac{1}{2} \\ &= -1. \end{aligned}$$

Hence the frame is negatively oriented.

The isometry found in the earlier exercise has the following matrix for its orthogonal part.

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}$$

The determinant of this matrix is

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left(-\frac{1}{9\sqrt{2}} - \frac{8}{9\sqrt{2}}\right) - 0 + \frac{1}{\sqrt{2}}\left(-\frac{8}{9\sqrt{2}} - \frac{1}{9\sqrt{2}}\right) \\ &= -\frac{1}{2} - \frac{1}{2} \\ &= -1. \end{aligned}$$

This fits in with the calculations of the frames' orientations. Since the isometry carries a positively oriented frame to a negatively oriented one, its sign must be -1.

Solution 3.2

Firstly, we have

$$\mathbf{v} \times \mathbf{w} = (-2, 0, -6).$$

Now we calculate the various images.

$$\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix};$$

$$\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{11}{3} \\ -\frac{7}{3} \end{pmatrix};$$

$$\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ \frac{8}{3} \\ -\frac{14}{3} \end{pmatrix}.$$

Since the isometry defined by C has orthogonal part C, we have

$$C_* = C$$

and, so,

$$\operatorname{sgn} C_* = \det C.$$

Now, calculating the determinant of C shows that

$$\operatorname{sgn} C_* = -1.$$

Also, we have

$$\begin{aligned} C(\mathbf{v}) \times C(\mathbf{w}) &= \left(-\frac{10}{3}, -\frac{8}{3}, \frac{14}{3}\right) \\ &= -\left(\frac{10}{3}, \frac{8}{3}, -\frac{14}{3}\right) \\ &= -C(\mathbf{v} \times \mathbf{w}). \end{aligned}$$

Since $\operatorname{sgn} C_* = -1$, we have

$$\begin{aligned} C(\mathbf{v} \times \mathbf{w}) &= -C(\mathbf{v}) \times C(\mathbf{w}) \\ &= \operatorname{sgn} C_* C(\mathbf{v}) \times C(\mathbf{w}). \end{aligned}$$

This completes the check.

Solution 4.1

Our main weapon for tackling such a question is Theorem 4.2. If we set

$$\gamma = F(\beta),$$

then we have exactly the conditions specified in Theorem 4.2 and we may apply its conclusions.

(a) If β is a cylindrical helix, we know that

$$\frac{\tau_\beta}{\kappa_\beta} = a,$$

where a is a constant. But

$$\begin{aligned} \frac{\tau_\gamma}{\kappa_\gamma} &= \frac{\pm\tau_\beta}{\kappa_\beta} \\ &= \pm a, \end{aligned}$$

which is also a constant. Thus $\gamma = F(\beta)$ is also a cylindrical helix.

(b) Here we apply the definition of spherical image. The coordinate functions of $\tilde{\beta}$ are those of the unit tangent β'. Abusing notation slightly, we can write

$$\tilde{\beta} = \beta'.$$

Similarly,

$$\tilde{\gamma} = \gamma'.$$

Now we bring in the isometry F and its derivative map C.

$$\begin{aligned} \tilde{\gamma} &= \gamma' \\ &= (F(\beta))' \\ &= F_*(\beta') \\ &= C(\beta') \\ &= C(\tilde{\beta}). \end{aligned}$$

This is exactly what we were asked to show.

There is an alternative proof that you may prefer. We can write the spherical image of β as

$$\tilde{\beta} = T,$$

(with much the same sort of abuse of notation as above). Similarly, the spherical image of $F(\beta)$ is

$$\begin{aligned} \overline{T} &= F_*(T) \quad \text{(using Theorem 4.2)} \\ &= F_*(\tilde{\beta}) \\ &= C(\tilde{\beta}). \end{aligned}$$

Solution 4.2

In order to find $\overline{\alpha}$, we have to calculate

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 2t \end{pmatrix} = \begin{pmatrix} -\cos t \\ \frac{1}{\sqrt{2}}(\sin t - 2t) \\ \frac{1}{\sqrt{2}}(\sin t + 2t) \end{pmatrix},$$

Thus

$$\overline{\alpha}(t) = \left(-\cos t, \frac{1}{\sqrt{2}}(\sin t - 2t), \frac{1}{\sqrt{2}}(\sin t + 2t)\right).$$

Since, with our usual mild abuse of notation, $C_* = C$, we can calculate $C_*(Y)$ as follows.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} t \\ 1 - t^2 \\ 1 + t^2 \end{pmatrix} = \begin{pmatrix} -t \\ -\sqrt{2}t^2 \\ \sqrt{2} \end{pmatrix},$$

so

$$\overline{Y} = C_*(Y) = (-t, -\sqrt{2}t^2, \sqrt{2}).$$

Now,

$$Y' = (1, -2t, 2t),$$

$$\overline{Y}' = (-1, -2\sqrt{2}t, 0)$$

and since

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix} = \begin{pmatrix} -1 \\ -2\sqrt{2}t \\ 0 \end{pmatrix},$$

we have

$$\begin{aligned} C_*(Y') &= (-1, -2\sqrt{2}t, 0) \\ &= \overline{Y}'. \end{aligned}$$

We have

$$\alpha'(t) = (-\sin t, \cos t, 2),$$

$$\alpha''(t) = (-\cos t, -\sin t, 0),$$

$$\overline{\alpha}'(t) = \left(\sin t, \frac{1}{\sqrt{2}}(\cos t - 2), \frac{1}{\sqrt{2}}(\cos t + 2)\right),$$

$$\overline{\alpha}''(t) = \left(\cos t, -\frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\sin t\right).$$

Now

$$\begin{aligned} C(\alpha'') &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t \\ -\frac{1}{\sqrt{2}}\sin t \\ -\frac{1}{\sqrt{2}}\sin t \end{pmatrix} \end{aligned}$$

so

$$C_*(\alpha'') = \overline{\alpha}''.$$

Finally,

$$\begin{aligned} Y' \cdot \alpha'' &= (1, -2t, 2t) \cdot (-\cos t, -\sin t, 0) \\ &= -\cos t + 2t \sin t. \\ \overline{Y}' \cdot \overline{\alpha}'' &= (-1, -2\sqrt{2}t, 0) \cdot \left(\cos t, -\frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{2}} \sin t\right) \\ &= -\cos t + 2t \sin t \\ &= Y' \cdot \alpha''. \end{aligned}$$

Solution 5.1

In order to invoke our main theorem, we must find the curvature and torsion functions of both curves. They are actually unit-speed curves, but there is no harm in using the more general methods usually used for non unit-speed curves. We choose to use the unit-speed methods.

We begin with α.

$$\alpha'(t) = (-\sin t, \cos t, 0).$$

From this we can see that α is unit speed, since

$$\|\alpha'(t)\|^2 = \sin^2 t + \cos^2 t = 1.$$

Hence

$$\begin{aligned} T_\alpha &= (-\sin t, \cos t, 0); \\ T'_\alpha &= (-\cos t, -\sin t, 0), \\ \kappa_\alpha &= \|T'_\alpha\| \\ &= 1; \\ N_\alpha &= T'_\alpha / \kappa_\alpha \\ &= (-\cos t, -\sin t, 0); \\ B_\alpha &= T_\alpha \times N_\alpha \\ &= (0, 0, 1); \\ B'_\alpha &= (0, 0, 0), \\ \tau_\alpha &= -B'_\alpha \cdot N_\alpha \\ &= 0. \end{aligned}$$

Next, we deal with β.

$$\beta'(t) = \frac{1}{3}(2 \sin t + 2 \cos t, -2 \sin t + \cos t, -\sin t + 2 \cos t).$$

$$\begin{aligned} \|\beta'(t)\|^2 &= \frac{1}{9}(4 \sin^2 t + 4 \cos^2 t + 4 \sin^2 t + \cos^2 t \\ &\quad + \sin^2 t + 4 \cos^2 t) \\ &= 1. \end{aligned}$$

Thus β is also unit speed.

$$\begin{aligned} T_\beta &= \frac{1}{3}(2 \sin t + 2 \cos t, -2 \sin t + \cos t, -\sin t + 2 \cos t); \\ T'_\beta &= \frac{1}{3}(2 \cos t - 2 \sin t, -2 \cos t - \sin t, -\cos t - 2 \sin t), \\ \kappa_\beta &= \|T'_\beta\| \\ &= \frac{1}{3}\sqrt{9 \cos^2 t + 9 \sin^2 t} \\ &= 1; \\ N_\beta &= T'_\beta / \kappa_\beta \\ &= \frac{1}{3}(2 \cos t - 2 \sin t, -2 \cos t - \sin t, -\cos t - 2 \sin t); \\ B_\beta &= T_\beta \times N_\beta \\ &= \frac{1}{9}(3, 6, -6) \\ &= \frac{1}{3}(1, 2, -2); \\ B'_\beta &= 0, \\ \tau_\beta &= -B'_\beta \cdot N_\beta \\ &= 0. \end{aligned}$$

From the above we can see that the two curves satisfy the requirements of the main theorem and are, therefore, congruent.

To find the isometry, we calculate the Frenet frame at $t = 0$ for both curves.

For α:

$$\begin{aligned} T_\alpha(0) &= (0, 1, 0), \\ N_\alpha(0) &= (-1, 0, 0), \\ B_\alpha(0) &= (0, 0, 1). \end{aligned}$$

For β:

$$\begin{aligned} T_\beta(0) &= \frac{1}{3}(2, 1, 2), \\ N_\beta(0) &= \frac{1}{3}(2, -2, -1), \\ B_\beta(0) &= \frac{1}{3}(1, 2, -2). \end{aligned}$$

Since the torsions of the two curves are *equal*, we require an isometry that will carry the Frenet frame at $\alpha(0)$ to the Frenet frame at $\beta(0)$, and also map $\alpha(0)$ to $\beta(0)$. The attitude matrices of the Frenet frames are

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

respectively.

If we apply the ‘‘$^tB\,A$’ formula from Section 2, then the orthogonal part of the isometry is

$$\begin{aligned} C &= \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}. \end{aligned}$$

We also have

$$\begin{aligned} \alpha(0) &= (1, 0, 0), \\ \beta(0) &= \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right). \end{aligned}$$

Next we calculate $C(\alpha(0))$.

$$\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

$$C(\alpha(0) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

The translation part of the isometry must be by the vector

$$\beta(0) - C(\alpha(0)) = \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$
$$= (1, -1, 0).$$

Since the orthogonal and translation parts have been found, the isometry is completely specified. We can give a description as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Solution 5.2

We use the hint offered and give only the calculations for β. Since the hint implies that neither is unit speed, we use the general formulas.

First we calculate speed, curvature and torsion.

$$\beta'(t) = (3\sqrt{2}, -6t, 3\sqrt{2}t^2),$$
$$\beta''(t) = (0, -6, 6\sqrt{2}t),$$
$$\beta'''(t) = (0, 0, 6\sqrt{2}).$$

$$\beta'(t) \times \beta''(t) = (-18\sqrt{2}t^2, -36t, -18\sqrt{2})$$
$$= -18\sqrt{2}(t^2, \sqrt{2}t, 1).$$

$$\beta'(t) \times \beta''(t) \cdot \beta'''(t) = -216.$$

$$v_\beta = \|\beta'(t)\|$$
$$= \sqrt{18 + 36t^2 + 18t^4}$$
$$= \sqrt{18}\sqrt{1 + 2t^2 + t^4}$$
$$= 3\sqrt{2}(1 + t^2).$$

$$\|\beta'(t) \times \beta''(t)\| = 18\sqrt{2}\sqrt{1 + 2t^2 + t^4}$$
$$= 18\sqrt{2}(1 + t^2).$$

$$\kappa_\beta = \frac{\|\beta'(t) \times \beta''(t)\|}{v_\beta^3}$$
$$= \frac{18\sqrt{2}(1 + t^2)}{54\sqrt{2}(1 + t^2)^3}$$
$$= \frac{1}{3(1 + t^2)^2}.$$

$$\tau_\beta = \frac{\beta'(t) \times \beta''(t) \cdot \beta'''(t)}{\|\beta'(t) \times \beta''(t)\|^2}$$
$$= \frac{-216}{18^2 \times 2 \times (1 + t^2)^2}$$
$$= -\frac{1}{3(1 + t^2)^2}.$$

Comparing these with the results found by *O'Neill*, we see that

$$v_\alpha = v_\beta > 0,\ \kappa_\alpha = \kappa_\beta,\ \tau_\alpha = -\tau_\beta.$$

Thus the curves are congruent.

To find the isometry we require the Frenet frames at $t = 0$. There is little point in finding the general forms of the Frenet frame for β, we might as well just calculate the values of the expressions required by Chapter II, Theorem 4.3 at $t = 0$ and use them to find the frame at this point only.

For α we have the results from Chapter II available and they yield the following.

$$T_\alpha(0) = \frac{1}{\sqrt{2}}(1, 0, 1),$$
$$N_\alpha(0) = (0, 1, 0),$$
$$B_\alpha(0) = \frac{1}{\sqrt{2}}(-1, 0, 1).$$

For β we proceed as indicated above, using the results already obtained.

$$T_\beta(0) = \frac{\beta'(0)}{v_\beta(0)}$$
$$= \frac{1}{3\sqrt{2}} 3\sqrt{2}(1, 0, 0)$$
$$= (1, 0, 0);$$

$$B_\beta(0) = \frac{\beta'(0) \times \beta''(0)}{\|\beta'(0) \times \beta''(0)\|}$$
$$= \frac{1}{18\sqrt{2}} 18(0, 0, -\sqrt{2})$$
$$= (0, 0, -1);$$

$$N_\beta(0) = B_\beta(0) \times T_\beta(0)$$
$$= (0, -1, 0).$$

Because $\tau_\alpha = -\tau_\beta$, we want

$$\left.\begin{matrix} T_\alpha(0) \\ N_\alpha(0) \\ B_\alpha(0) \end{matrix}\right\} \longmapsto \left\{\begin{matrix} T_\beta(0) \\ N_\beta(0) \\ -B_\beta(0) \end{matrix}\right. .$$

(Note the minus sign.)

The respective attitude matrices of these frames are

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

since $-B_\beta(0) = (0, 0, 1)$.

The orthogonal part C of the isometry is, therefore, given by

$$C = {}^tB\,A$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

It remains to find only the translation part. We have

$$\alpha(0) = \beta(0) = (0, 0, 0).$$

Since

$$C(\alpha(0)) = (0, 0, 0) = \beta(0),$$

the translation part is the identity and the isometry is just the orthogonal transformation defined by the matrix C.

Solution 5.3

At first sight the condition given in the question looks somewhat mysterious. Let us begin by looking at the first half of the following assertion.

If α and β are congruent, then β can be expressed as indicated.

We start by writing down what we know. Using the usual notation,

$$\alpha(t) = \alpha_1(t)(1,0,0) + \alpha_2(t)(0,1,0) + \alpha_3(t)(0,0,1).$$

We also know that there is an isometry $F = TC$ such that

$$\beta = F(\alpha).$$

If we set

$$\begin{aligned}\mathbf{e}_1 &= C(1,0,0),\\ \mathbf{e}_2 &= C(0,1,0),\\ \mathbf{e}_3 &= C(0,0,1),\end{aligned}$$

then, by the linearity of the orthogonal part,

$$C(\alpha) = \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2 + \alpha_3(t)\mathbf{e}_3.$$

Since the $\mathbf{e}_i$s are images of an orthonormal basis under an orthogonal transformation, it follows that they are also orthonormal, that is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}.$$

If we define $\mathbf{p}$ to be $T(0,0,0)$, then

$$\begin{aligned}\beta &= F(\alpha)\\ &= \mathbf{p} + C(\alpha)\\ &= \mathbf{p} + \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2 + \alpha_3(t)\mathbf{e}_3.\end{aligned}$$

This completes the proof of the first half.

The first half of the proof also indicates how to prove the other half. Suppose now that β can be written as

$$\beta = \mathbf{p} + \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2 + \alpha_3(t)\mathbf{e}_3.$$

Let T be translation by $\mathbf{p}$ and let C be the (unique) orthogonal transformation that maps

$$\left.\begin{matrix}(1,0,0)\\(0,1,0)\\(0,0,1)\end{matrix}\right\} \longmapsto \left\{\begin{matrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{matrix}\right. .$$

Then, from the way we have defined F and the linearity of C,

$$\beta = F(\alpha)$$

and the curves are congruent.

Solution 5.4

We begin by calculating the curvature and torsion of β.

$$\begin{aligned}\beta'(t) &= (1+\sqrt{3}\cos t, -2\sin t, \sqrt{3} - \cos t),\\ \beta''(t) &= (-\sqrt{3}\sin t, -2\cos t, \sin t),\\ \beta'''(t) &= (-\sqrt{3}\cos t, 2\sin t, \cos t);\end{aligned}$$

$$\beta'(t) \times \beta''(t) = (2\sqrt{3}\cos t - 2, -4\sin t, -2\sqrt{3} - 2\cos t),$$

$$\beta'(t) \times \beta''(t) \cdot \beta'''(t) = -8;$$

$$\begin{aligned}v_\beta &= \|\beta'(t)\|\\ &= \sqrt{4+4}\\ &= 2\sqrt{2};\end{aligned}$$

$$\begin{aligned}\kappa_\beta &= \frac{\|\beta'(t) \times \beta''(t)\|}{v_\beta^3}\\ &= \frac{4\sqrt{2}}{16\sqrt{2}}\\ &= \frac{1}{4};\end{aligned}$$

$$\begin{aligned}\tau_\beta &= \frac{\beta'(t) \times \beta''(t) \cdot \beta'''(t)}{\|\beta'(t) \times \beta''(t)\|^2}\\ &= \frac{-8}{32}\\ &= -\frac{1}{4}.\end{aligned}$$

Since both the curvature and torsion are constant, so is their ratio and β is a cylindrical helix. Because the curvature is constant, it is a circular helix.

In order to find the values of a and b that will make α congruent to β, we calculate the speed, curvature and torsion for α.

$$\begin{aligned}\alpha'(t) &= (-a\sin t, a\cos t, b),\\ \alpha''(t) &= (-a\cos t, -a\sin t, 0),\\ \alpha'''(t) &= (a\sin t, -a\cos t, 0),\end{aligned}$$

$$\alpha'(t) \times \alpha''(t) = (ab\sin t, -ab\cos t, a^2),$$

$$\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t) = a^2 b.$$

$$\begin{aligned}\mathbf{v}_\alpha &= \|\alpha'(t)\|\\ &= \sqrt{a^2+b^2},\end{aligned}$$

$$\begin{aligned}\kappa_\alpha &= \frac{\|\alpha'(t) \times \alpha''(t)\|}{v_\alpha^3}\\ &= \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)\sqrt{a^2+b^2}}\\ &= \frac{a}{a^2+b^2};\end{aligned}$$

$$\begin{aligned}\tau_\alpha &= \frac{\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2}\\ &= \frac{a^2 b}{a^2(a^2+b^2)}\\ &= \frac{b}{a^2+b^2}.\end{aligned}$$

We require

$$\begin{aligned}v_\alpha &= v_\beta\\ \kappa_\alpha &= \kappa_\beta.\end{aligned}$$

The first requirement forces

$$a^2 + b^2 = 8$$

and so the second then gives

$$a = 2.$$

It follows that

$$b^2 = 4.$$

Either of the solutions for b would be acceptable, taking

$$b = -2$$

actually gives

$$\tau_\alpha = \tau_\beta,$$

which makes finding the isometry slightly easier. Note that this ambiguity is to be expected, depending on whether we require α to be a helix of the same 'handedness' as β or not.

With the choice of b as above, we can now find the Frenet frames at $t = 0$ and, hence, the isometry.

For β we have

$$\begin{aligned}\beta'(0) &= (1+\sqrt{3}, 0, \sqrt{3}-1),\\ T_\beta(0) &= \frac{1}{2\sqrt{2}}(1+\sqrt{3}, 0, \sqrt{3}-1),\end{aligned}$$

$$\begin{aligned}
B_\beta(0) &= \frac{\beta'(0)\times\beta''(0)}{\|\beta'(0)\times\beta''(0)\|}\\
&= \frac{1}{2\sqrt{2}}(\sqrt{3}-1, 0, -\sqrt{3}-1),\\
N_\beta(0) &= B_\beta(0)\times T_\beta(0)\\
&= \frac{1}{8}(0,-8,0)\\
&= (0,-1,0).
\end{aligned}$$

The corresponding results for α are

$$\begin{aligned}
\alpha'(0) &= (0,-2,0),\\
T_\alpha(0) &= \frac{1}{\sqrt{2}}(0,1,-1),\\
B_\alpha(0) &= \frac{\alpha'(0)\times\alpha''(0)}{\|\alpha'(0)\times\alpha''(0)\|}\\
&= \frac{1}{\sqrt{2}}(0,1,1),\\
N_\alpha(0) &= B_\alpha(0)\times T_\alpha(0)\\
&= (-1,0,0).
\end{aligned}$$

If we let $F = TC$ be the isometry such that

$$F(\alpha) = \beta,$$

then the attitude matrices of the relevant frames are

$$A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 & \frac{\sqrt{3}-1}{2\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 & \frac{-\sqrt{3}-1}{2\sqrt{2}} \end{pmatrix}.$$

Note that there is no sign reversal because the torsions are equal.

The orthogonal part of the isometry is given by $C = {}^tB\,A$, noting that, in this case, ${}^tB = B$.

$$\begin{aligned}
C &= {}^tB\,A\\
&= \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 & \frac{\sqrt{3}-1}{2\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 & \frac{-\sqrt{3}-1}{2\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\\
&= \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.
\end{aligned}$$

We also have

$$\begin{aligned}
\alpha(0) &= (2,0,0),\\
\beta(0) &= (0,2,0).
\end{aligned}$$

Since

$$C(\alpha(0)) = (0,2,0) = \beta(0)$$

the translation part is the identity. Thus F is the orthogonal transformation defined by C.

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